Topic 13 – Euler Method

Sometimes you don't have a method to find an exact solution to a differential equation. So instead you approximate the solution.

One method to do this is called Euler's method.
Its used for first-order initial-value problems:

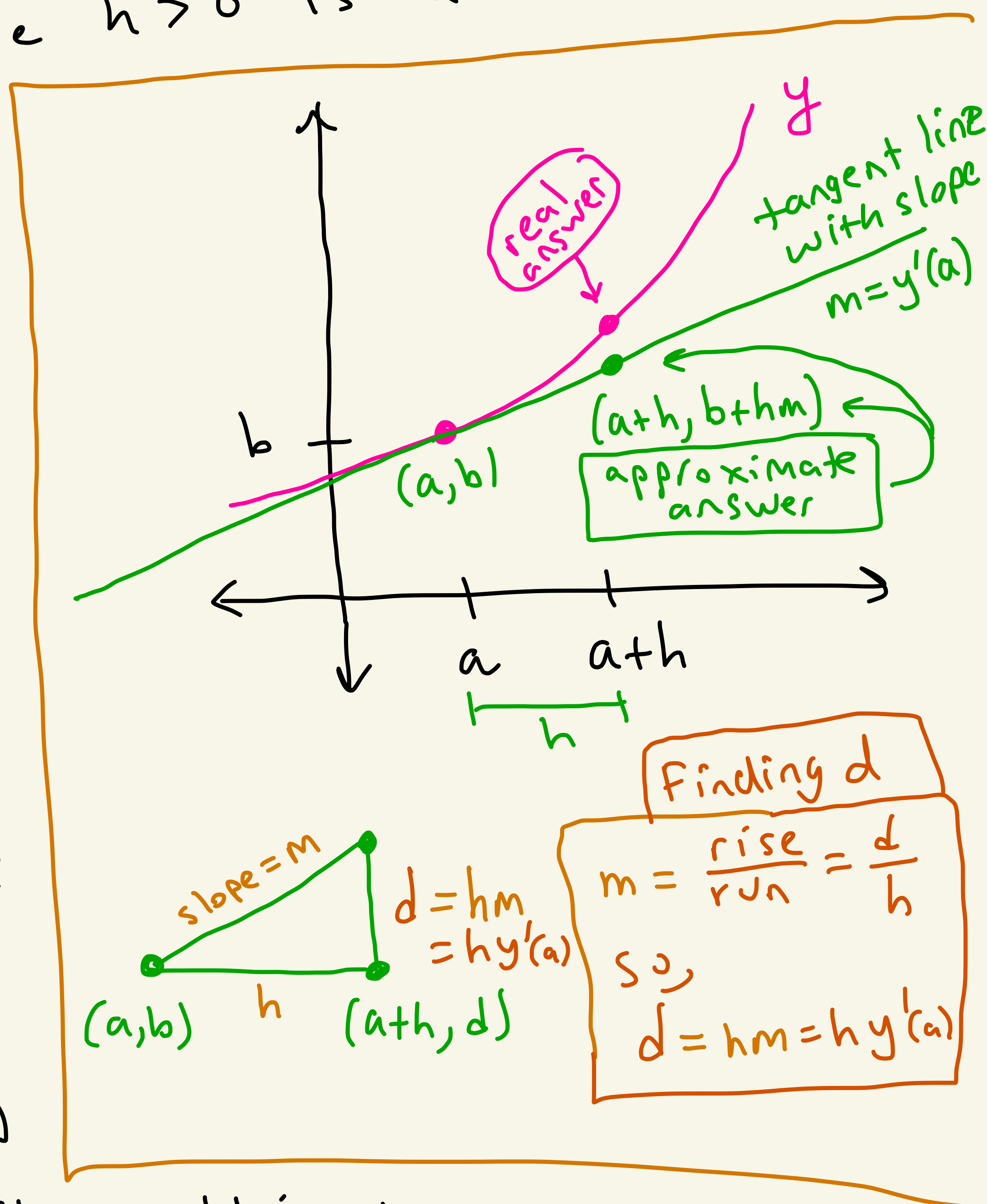
$$y' = f(x, y)$$
$$y(x_0) = y_0$$

The idea goes like this: Suppose you know that a function y satisfies $y(a)=b$ and $y'(a)=m$.

Then suppose you want to approximate $y(a+h)$ where $h>0$ is a small number.

If you go h distance along the x-axis and follow the tangent line then the y-value will be

$$y = b + hm = b + hy'(a).$$



You just iterate this over and over to get your approximation.

How to do this?

Suppose we want to approximate a solution to

$$y' = f(x,y)$$

$$y(x_0) = y_0$$

this tells us the slope of the solution at any point

this gives us an actual value of a solution at some starting point x_0

Pick some value $h > 0$.

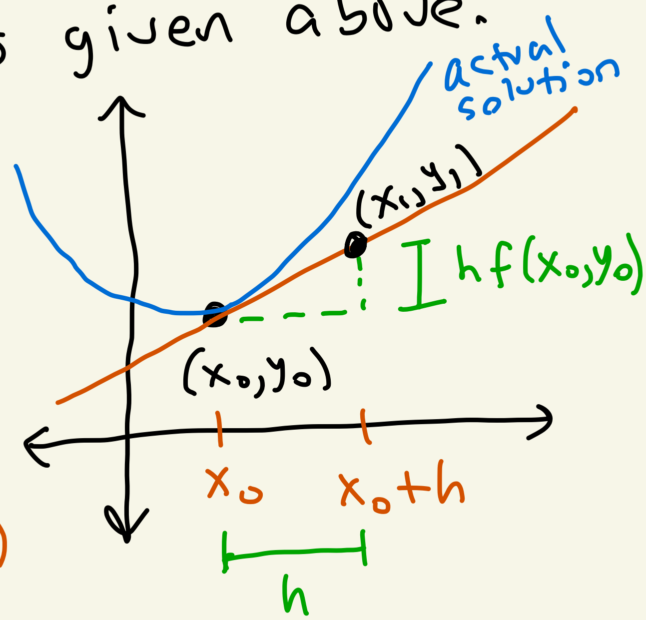
[The smaller the h the better approximation you will get.]

Our starting point (x_0, y_0) is given above.

Set

$$x_1 = x_0 + h$$

$$y_1 = y_0 + h \cdot \underbrace{f(x_0, y_0)}_{\substack{\text{this is the slope} \\ y' \text{ at } (x_0,y_0) \\ \text{of a solution}}}$$



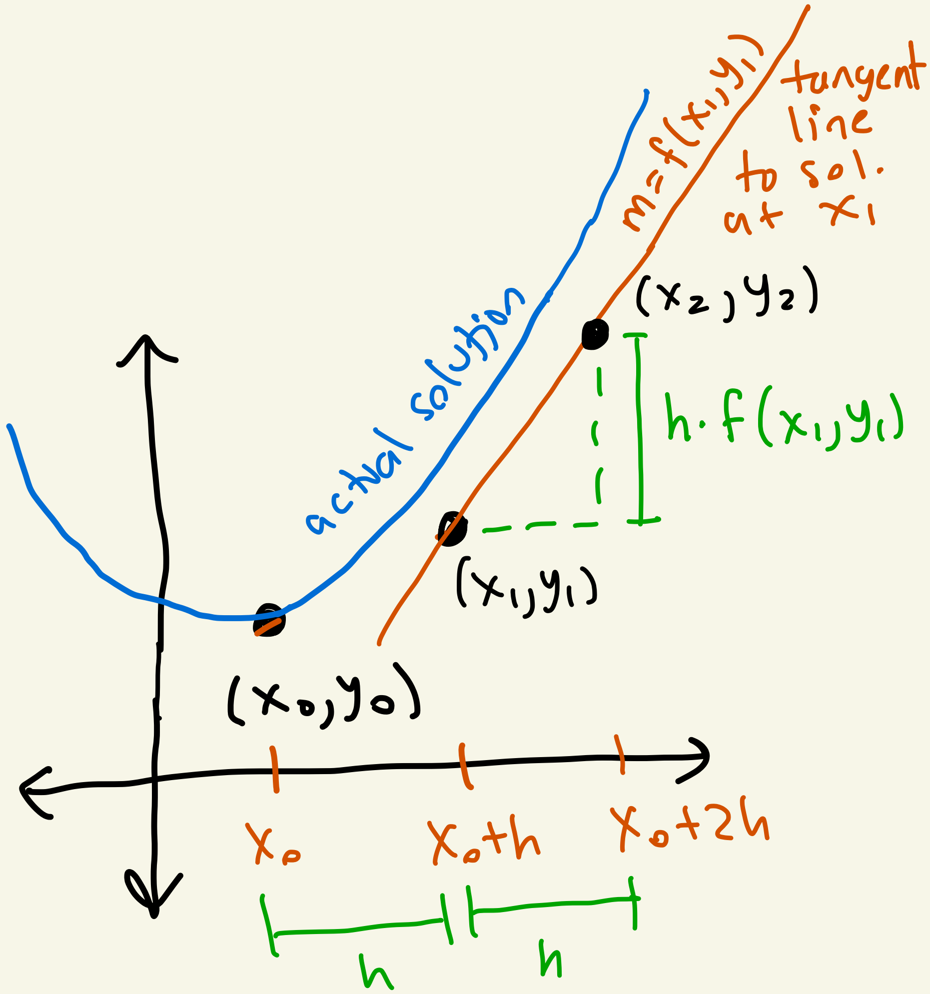
Now let's approximate the next point.

Set

$x_2 = x_1 + h$

$y_2 = y_1 + h f(x_1, y_1)$

So, (x_2, y_2) gives an approximation to the solution at x_2 of y_2.



Keep iterating this idea to get Euler's method.

Euler's Method

Suppose we want to approximate a solution to

$$y' = f(x, y), \quad y(x_0) = y_0$$

Pick some $h > 0$.

We are given the starting point (x_0, y_0) above.

Then set

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$$

for $n \geq 1$

Ex:

Consider the initial-value problem

$$\boxed{\begin{aligned} y' &= xy \\ y(0) &= 1 \end{aligned}}$$

$$\boxed{\begin{aligned} &y' - xy = 0 \\ &e^{-\frac{1}{2}x^2} y' - x e^{-\frac{1}{2}x^2} y = 0 \\ &\left(y e^{-\frac{1}{2}x^2}\right)' = 0 \\ &y e^{-\frac{1}{2}x^2} = C \\ &y = C e^{\frac{1}{2}x^2} \\ &y(0) = 1 \to C = 1 \\ &y = e^{\frac{1}{2}x^2} \end{aligned}}$$

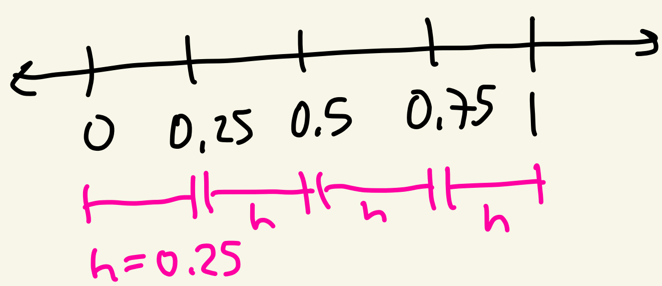
From earlier methods we know the solution is $y = e^{\frac{1}{2}x^2}$.

Let's pretend that we don't know this.

Let's try to approximate the solution when $0 \le x \le 1$.

First let's divide up $0 \le x \le 1$ into smaller segments.

Let $h = 0.25 = \frac{1-0}{4}$.



Using h we can

break $0 \le x \le 1$ into 4 equally sized segments. We will approximate the solution to $y' = xy$, $y(0) = 1$ at these four points.

The formula $y' = xy$ tells us the slope of the tangent line of the solution at any point. We can use this to approximate the solution $y = e^{\frac{1}{2}x^2}$ without knowing the solution.

Use the initial-value $y(0) = 1$ to get the first point in our approximation.

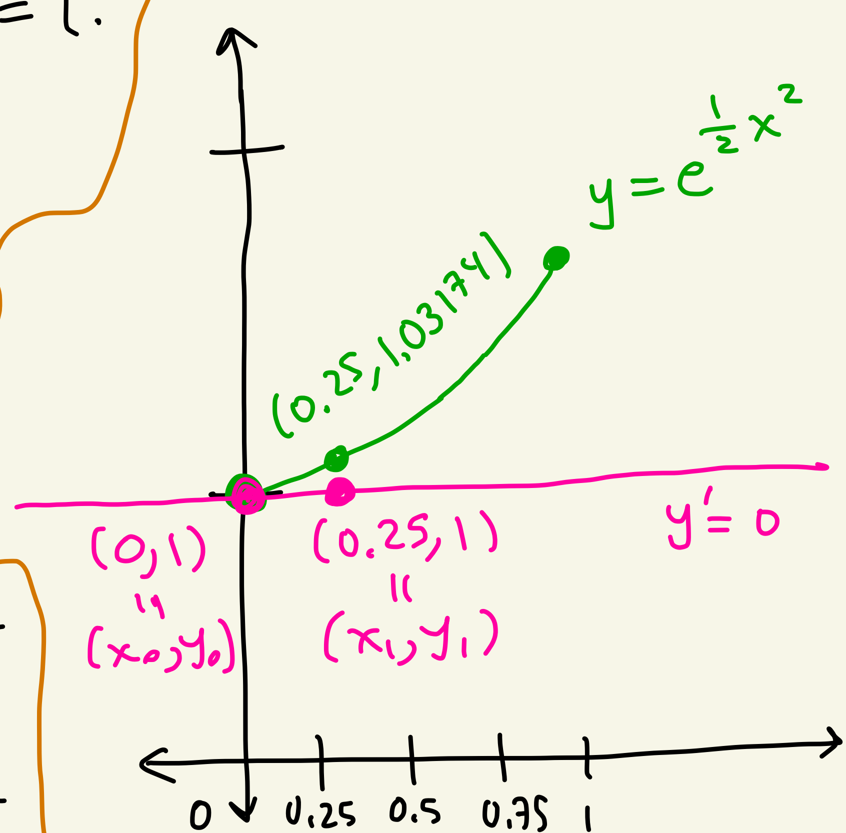
Let $x_0 = 0$, $y_0 = 1$.

The tangent line has slope

$y' = x_0 y_0 = 0$

at this point.

Move $h = 0.25$ along the tangent line to get the next approximate



point (x_1, y_1). This is:

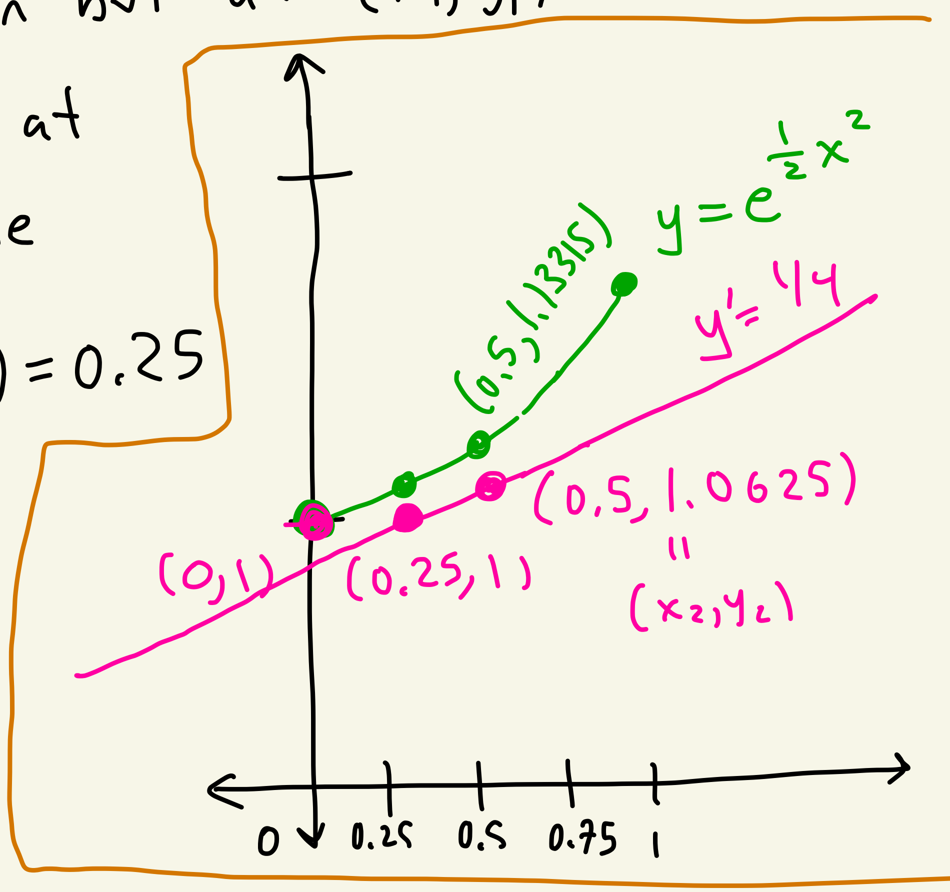
$$x_1 = x_0 + h = 0 + 0.25 = 0.25$$

$$y_1 = y_0 + h \cdot \underbrace{y'(x_0, y_0)}_{x_0 y_0}$$

$$= 1 + 0.25(0)(1)$$

$$= 1$$

Now do this again but at $(x_1, y_1) = (0.25, 1)$



The tangent line at $(0.25, 1)$ has slope

$$y'(0.25, 1) = (0.25)(1) = 0.25$$

$\boxed{y' = xy}$

Set the next approximate point to be

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$

$$y_2 = y_1 + h \cdot y'(x_1, y_1)$$

$$= 1 + 0.25(0.25)(1) = 1.0625$$

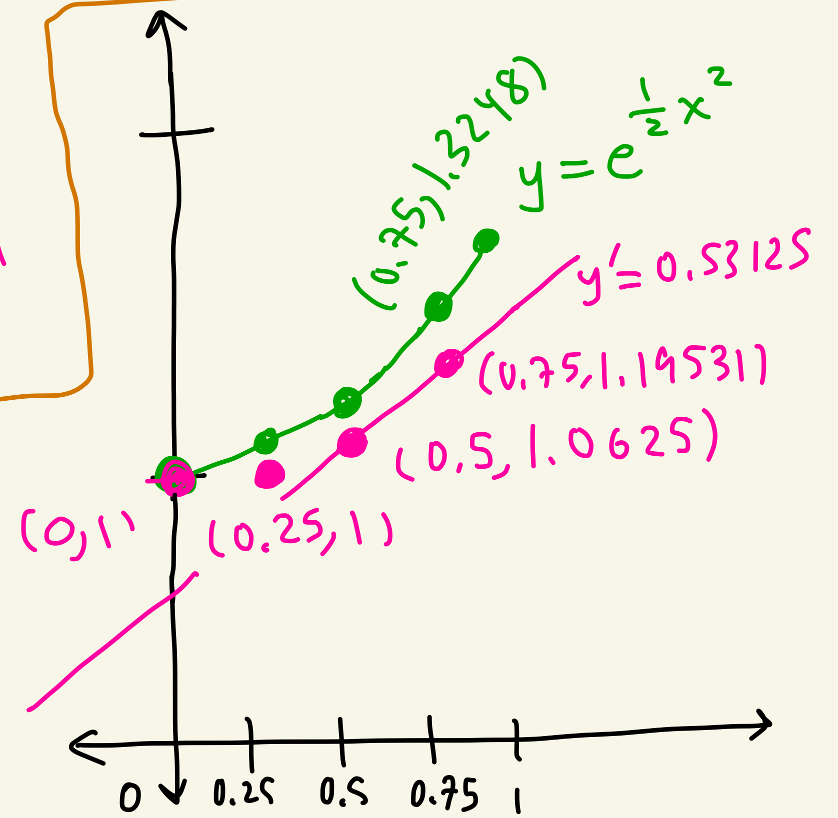
Keep going...

$$x_3 = x_2 + h = 0.5 + 0.25 = 0.75$$

$$y_3 = y_2 + h\,y'(x_2, y_2)$$

$$= 1.0625 + 0.25\overbrace{(0.5)(1.0625)}^{y' = 0.53125}$$

$$= 1.19531$$



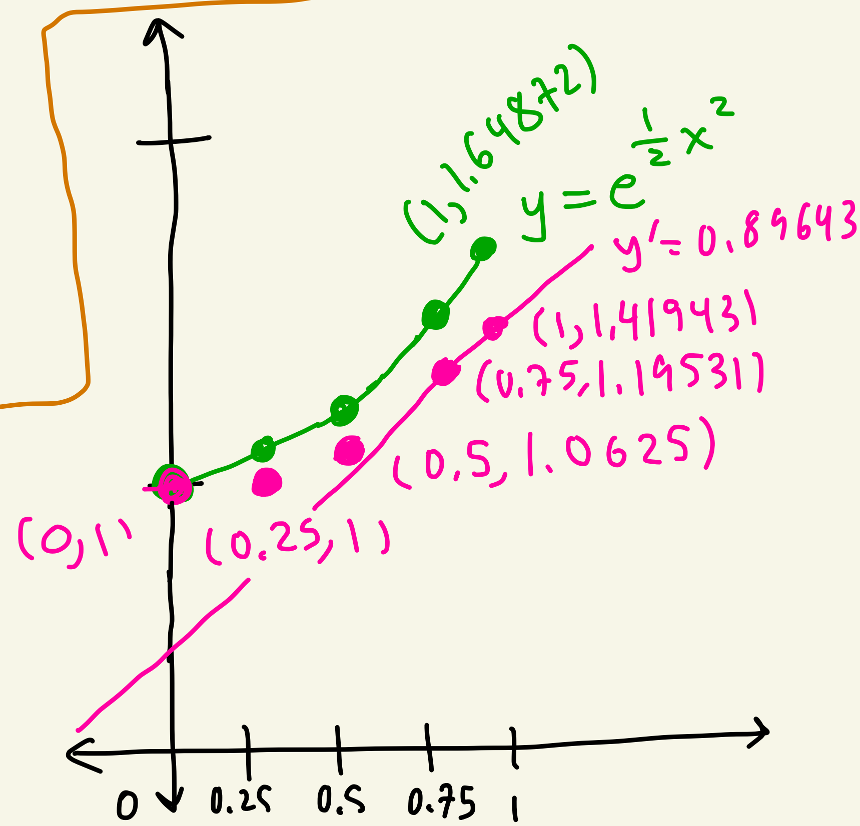
Last point...

$$x_4 = x_3 + h = 0.75 + 0.25 = 1$$

$$y_4 = y_3 + h\,y'(x_3, y_3)$$

$$= 1.19531 + 0.25\overbrace{(0.75)(1.19531)}^{y' = 0.896483}$$

$$= 1.41943$$



Summary

n	x_n	y_n	actual solution $y=e^{\frac{1}{2}x^2}$ evaluated at x_n
0	0	1	1
1	0.25	1	1.03174
2	0.5	1.0625	1.13315
3	0.75	1.19531	1.32478
4	1	1.41943	1.64872

If we used a smaller h we would get a way better approximation. This example is to give an idea of how the method works.

Ex: Approximate a solution to

$$\boxed{\begin{aligned} y' &= xy \\ y(0) &= 1 \end{aligned}} \leftarrow \begin{aligned} f(x,y) &= xy \\ x_0 = 0,\ y_0 &= 1 \end{aligned}$$

on the interval $0 \le x \le 0.5$ using $h = 0.1$

The Euler equations here are

$$\boxed{\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + h \cdot f(x_{n-1}, y_{n-1}) \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} &\left.\begin{aligned} x_n &= x_{n-1} + 0.1 \\ y_n &= y_{n-1} + (0.1)\, x_{n-1}\, y_{n-1} \end{aligned}\right\} n \geq 1 \\ &x_0 = 0 \\ &y_0 = 1 \end{aligned}}$$

For $n=1$:

$$\boxed{\begin{aligned} x_1 &= x_0 + h = 0 + 0.1 = 0.1 \\ y_1 &= y_0 + h \cdot x_0 \cdot y_0 \\ &= 1 + (0.1)(0)(1) \\ &= 1 \end{aligned}} \leftarrow \boxed{\begin{aligned} x_1 &= 0.1 \\ y_1 &= 1 \end{aligned}}$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$
$$y_2 = y_1 + h \cdot x_1 \cdot y_1$$
$$= 1 + (0.1)(0.1)(1)$$
$$= 1.01$$

$$x_2 = 0.2$$
$$y_2 = 1.01$$

$$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$$
$$y_3 = y_2 + h \cdot x_2 \cdot y_2$$
$$= 1.01 + (0.1)(0.2)(1.01)$$
$$= 1.0302$$

$$x_3 = 0.3$$
$$y_3 = 1.0302$$

$$x_4 = x_3 + h = 0.4$$
$$y_4 = y_3 + h x_3 y_3$$
$$= 1.0302 + (0.1)(0.3)(1.0302)$$
$$= 1.061106$$

$$x_4 = 0.4$$
$$y_4 = 1.061106$$

$$x_5 = x_4 + h = 0.5$$
$$y_5 = y_4 + h x_4 y_4$$
$$= 1.061106 + (0.1)(0.4)(1.061106)$$
$$= 1.10355024$$

$$x_5 = 0.5$$
$$y_5 = 1.10355024$$

x_n	y_n	actual value of solution $e^{\frac{1}{2}x^2}$ at x_n	approximation we initinlly did with $h=0.25$
0	1	1	1
0.1	1	1.00501	
0.2	1.01	1.0202	
0.3	1.0302	1.04603	
0.4	1.061106	1.08329	
0.5	1.10355024	1.13315	1.0625

$h=0.25$
1